ELSEVIER
Discrete Applied Mathematics 115 (2001) 187–198
DISCRETE APPLIED MATHEMATICS

Rigidity of multi-story buildings

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Received 30 June 1999; received in revised form 31 January 2000; accepted 24 March 2000

Abstract

Bolker and Crapo gave a graph theoretical model of square grid frameworks with diagonal rods of certain squares. The problem of one-story buildings in special cases can be reduced to the planar problems. In this work the general case of one-story buildings will be considered and the results will be generalized to the case of multi-story buildings. © 2001 Published by Elsevier Science B.V.

Keywords: Grids; Rigidity; Frameworks; Graphs; Matroids

1. Preliminaries

Throughout, every framework consists of rigid rods and rotatable joints, first in the *plane*, and later—in the case of buildings—in the *three-dimensional space*, and *rigidity* stands for infinitesimal rigidity. (Some authors also use the term stability for infinitesimal rigidity.)

Firstly, let us consider a planar square grid. The elementary squares of the grid will be located by the numbers of its row and column. All rooms with the same row number or the same column number will be called an *X-corridor*, respectively, *Y-corridor*.

In any planar deformations of the grid the parallel rods of a corridor remain parallel. The common rotation of the rods will be called the *rotation of the corridor* (see Fig. 1). It is obvious that the deformations of the grid can be described with the signed measures of the rotations of corridors. A diagonal rod in a square of the grid indicates that the rotations of the corresponding X- and Y-corridors must be equal.

Let F be a framework composed of a $k \times l$ grid with certain additional rods forming diagonals of some squares. Its *graph* $G(F)$ is a bipartite graph with vertex set $V(G(F)) = A \cup B$ where $|A| = k$, $|B| = l$; the vertices correspond to the corridors in a

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PII: S0166-218X(01)00224-4

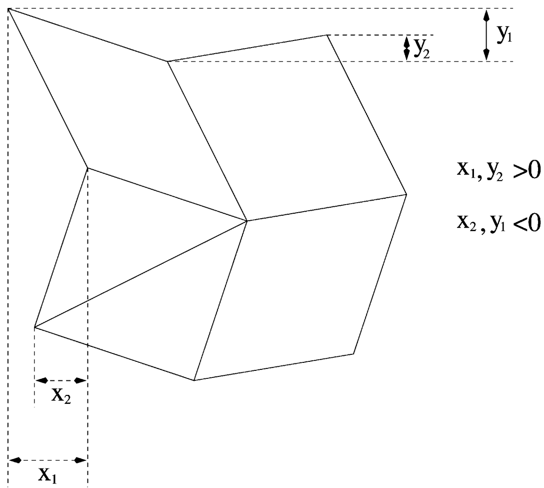

Fig. 1.

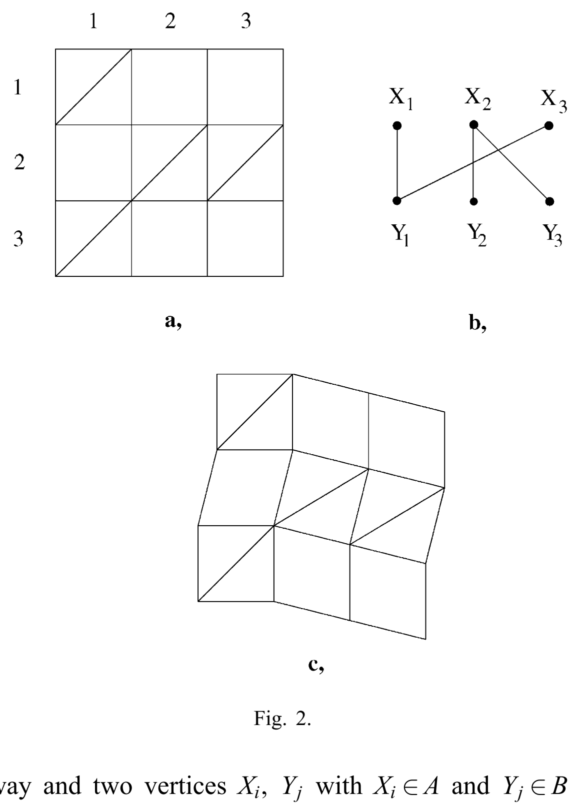

Fig. 2.

straightforward way and two vertices X_i, Y_j with $X_i \in A$ and $Y_j \in B$ are connected by an edge if and only if there is a diagonal in the square, formed by the intersection of X-corridor i and Y-corridor j. For example, Figs. 2a and b show a framework and its graph, respectively.

The main result of Bolker and Crapo [1] is the following:

Theorem 1.1. *Such a framework F is rigid if and only if its graph $G(F)$ is connected. Hence, a collection C of certain diagonals is a minimum set making F rigid if and only if $|C| = k + l - 1$ and the corresponding edges form a spanning tree of $G(F)$.*

For example, the graph of Fig. 2b is disconnected, hence the framework of Fig. 2a has deformations, like that of Fig. 2c.

Remark 1. Since a $k \times l$ grid, as a framework, clearly has $j = (k+1)(l+1)$ joints and $r = k(l+1) + l(k+1)$ rods and since a minimum rigid planar framework with j joints ($\geqslant 2$) must have $R = 2j - 3$ rods, the relation $|C| \geqslant R - r = k + l - 1$ is obvious. In the case of planar square grids this minimal number will hold. However, given a framework composed of a $k \times l$ square grid plus a system of $k + l - 1$ diagonals, the size of the rigidity matrix would be $O(kl)$, hence checking its rank with, say, Gaussian elimination would require $O(k^3 l^3)$ operations. By contrast, checking the condition of Theorem 1.1 requires $O(k + l)$ operations only. This linear time method can also be applied in the case of simple connected rectangular grids (without holes) [5].

The *one-story building* means that the joints of a $k \times l$ grid are connected to the ground by rods of uniform length. If each external vertical wall contains a diagonal rod then the problem of determining the rigidity of a one-story building reduces to the planar problem of determining whether the square grid is infinitesimally rigid if its corners are fixed to the plane.

Theorem 1.2 (Crapo [3]). *A framework of a one-story building which has rods in the external vertical walls is made rigid by certain diagonal rods if and only if $G(F)$ is connected or is an asymmetric 2-component graph where asymmetric means that*

$$\begin{vmatrix} |G_1 \cap A| & |G_1 \cap B| \\ |G_2 \cap A| & |G_2 \cap B| \end{vmatrix} \neq 0,$$

where G_1 and G_2 are the vertex sets of the two connected components of G, A and B are the two vertex sets of the bipartite graph.

So the minimum number of diagonal rods required is $k + l - 2$ (in addition to the 4 diagonals in the walls), which can be sufficient.

There are various possibilities of generalizing the basic problems. The first is using diagonal cables or struts (of some squares) to make the planar square grid or the one-story building (with rods in the external walls) rigid (see [1–3]). We (joint work with Zs. Gáspár and A. Recski) have considered the problem of long rods and cables [4] and the general problem of planar square grids with holes [5].

We would like to show the general case of one-story buildings with diagonal rods in Section 2, and then, in Section 3 its generalization to the case of t-story buildings.

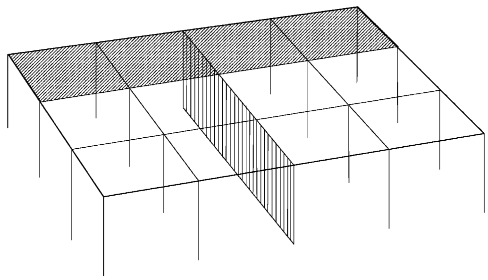

Fig. 3.

2. One-story buildings

Let us start with some notations. On the basis of the planar case the two sets of the horizontal corridors will be called X-*corridors* (X_i) and Y-*corridors* (Y_j), while the two sets of the vertical walls will be called X-*walls* (W_i^X) and Y-*walls* (W_j^Y), depending on their directions. Observing the building from above the X-corridors and Y-corridors as well as the X-walls and Y-walls will be numbered from top to bottom and from left to right, respectively. For example in Fig. 3 the first X-corridor (X_1) and the third Y-wall (W_3^Y) are striped.

Let us consider a one-story building with diagonal rods of some squares (vertical or horizontal). A diagonal rod in a (vertical) wall induces that this wall cannot move within its plane, so the joints of the wall can move only perpendicularly. If there are two crossing walls with diagonal rods in both, then the vertical rod in the intersection of the walls cannot move.

The first question is how many walls do we need to brace minimally apart from the bracing of the corridors? Let us suppose for a moment that the ceiling (formed by the corridors) is a rigid $k \times l$ rectangle. Now, if there is no diagonal rod in the walls our framework can fall down, and this cannot be prevented by bracing only one wall. Let us see how things stand if there are exactly two braced walls. If they are parallel the building can fall down perpendicular to the direction of the walls. So we need at least two perpendicular walls to be braced. But it is not sufficient: if we have only two braced walls in this way, the whole framework can turn infinitesimally around the rigid vertical rod in the intersection of the walls. So we can state the following

Statement 1. *If we want the building to be rigid we need at least three* (*vertical*) *walls to be braced and they must not all be parallel.*

The walls of a one-story building are *sufficiently braced* if and only if they satisfy the condition of Statement 1.

The next question is how can we describe the constraints caused by the diagonal rods in the walls and corridors. Let us consider a one-story building which has at least three braced walls and two among them are perpendicular. Now, we have at least two joints on the top of the vertical rods in the intersections of any two perpendicular pair of braced walls which cannot move, so that they are fixed. The joints are connected to the plane by vertical rods so that they can move from the original state in horizontal directions only. This means that preventing the horizontal motions of the joints (due to the fix joints) will make the building rigid. So we can claim the following

Statement 2. *If the walls are sufficiently braced and we prevent the horizontal motions of the joints* (*by means of the fixed joints*) *then the building will be rigid.*

Let us suppose that we have a one-story building which has vertical walls sufficiently braced. Then we have to prevent the deformation of the horizontal corridors as in a planar square grid. These motions of the building can be described with the rotations of the corridors (they will be denoted by x_i's and y_j's, respectively). If we put a diagonal rod into a horizontal square it indicates that the deformations of the corresponding X-corridor and Y-corridor must be equal.

About the vertical walls we can observe that the vertical rods in the intersections of any pair of perpendicular braced walls are fixed. So we can state that the sum of the deformations of X-corridors between any pair of braced X-walls must be equal to zero, and it is true similarly with Y-corridors and Y-walls. It was the main observation in the case of special one-story buildings in [3,2]. But using this *original system of equations* is a bit difficult because we cannot describe the effect of a wall-bracing itself. So we introduce *a new system of equations* to avoid this difficulty.

Firstly, we will introduce two new variables x_0 and y_0. Let us suppose that the braced walls are $W^X_{p_1},\dots,W^X_{p_m}$ and $W^Y_{q_1},\dots,W^Y_{q_n}$. Now our new system of equations consists of the following equations:

$$\sum_{i=0}^{p_{s-1}} x_i = 0 \quad s=1,\dots,m,$$

$$\sum_{j=0}^{q_{s-1}} y_j = 0 \quad s=1,\dots,n.$$

It is easy to see that the sets of solutions of the original system and the new system of equations, apart from the values of x_0 and y_0, are identical. On the one hand if we consider an original equation determined by a pair of braced X-walls (Y-walls) we can obtain it as the difference of the new equations corresponding to the two X-walls (Y-walls). On the other hand, the new equation corresponding to the first X-wall (Y-wall) defines the value of x_0 (y_0), so it is determined by the values of x_i's (y_j's), and the new equations corresponding to $W^X_{p_s}$ ($W^Y_{q_s}$) where $s \geqslant 2$ can be obtained as the sum of the defining equation of x_0 (y_0) and the original equation corresponding to $W^X_{p_1}$ and $W^X_{p_s}$ ($W^Y_{q_1}$ and $W^Y_{q_s}$).

Our way of reasoning leads to the following result:

Lemma 2.1. *Let us suppose that there is a one-story building B with walls sufficiently braced and with some diagonal rods of certain squares of corridors. Let us construct*

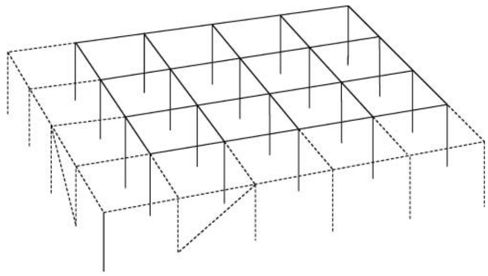

Fig. 4.

a new building $\bar{B}$ (see Fig. 4) *putting a new* X*-corridor* (X_0) *and a new* Y*-corridor* (Y_0) *to the building* B, *where the new walls are braced, so the vertical rod of the "new corner" in the intersection of the new walls is fixed. Now, the one-story building* B *is infinitesimally rigid if and only if the building* $\bar{B}$ *is infinitesimally rigid.*

Proof. The original equation-system of $\bar{B}$ is identical to the new equation-system of B. □

In the new system every diagonal rod (in a wall or in the ceiling) has a linear equation. The equation belonging to the diagonal rod in the ceiling in the intersection of the X-corridor X_i and the Y-corridor Y_j is

$$x_i - y_j = 0, \tag{1}$$

while the equation of the rod in the pth X-wall (on the basis of the extended building $\bar{B}$) is

$$\sum_{i=0}^{p-1} x_i = 0 \tag{2}$$

and the equation of the rod in the qth Y-wall is

$$\sum_{j=0}^{q-1} y_j = 0. \tag{3}$$

The building will be rigid if and only if the zero vector is the unique solution of the system of equations. So, if we have a $k \times l$ one-story building we can construct the coefficient matrix of this system where every row corresponds to a diagonal rod in the building. The condition of rigidity is that the rank of the matrix is $k + l + 2$ because there are so many variables in the equations. These observations imply the following

Theorem 2.2. *Let us consider a* $k \times l$ *one-story building with some diagonal rods of certain* (*horizontal or vertical*) *squares. The building is infinitesimally rigid if and only if the rank of the coefficient matrix of the corresponding system of equations* (*constructed from equations of types* (1)–(3)) *is* $k+l+2$. *So this is how many diagonal rods we need at least to make the building rigid.*

Remark 2. The minimal sets of diagonal rods which make the building rigid form the base set of a *representable matroid* whose representation is just the row vectors of the coefficient matrix of the linear equations. The two diagonal rods in the same square (but different directions) have the same effect, so we will define the matroid on the set of the horizontal and vertical squares to avoid the trivial parallel elements. On the other hand, the different squares of the same vertical wall will be parallel elements because their diagonal rods have the same effect on the movement of the building (and of course they have identical rows in the matrix).

Remark 3. Matroids are the most general structures where the so-called greedy algorithm can work. It means that we can state not only the existence of a minimal rigid system of diagonal rods in a one-story building but by declaring our preferences for the bracing of the squares in the building we can efficiently construct the most preferred rigid system of diagonal rods.

If we consider a set of diagonal rods only from the ceiling, its rank in the matroid can be at most $k+l-1$ (cf. the planar case). This also explains our original requirement that rigidity requires braces in at least three vertical walls.

We mentioned Crapo's result about the special case of the $(k \times l)$ one-story buildings (Theorem 1.2). Let us consider what happens if we apply the general theorem to this special case. The four equations obtained as the effect of the diagonal rods in the four external walls are

$$x_0 = y_0 = 0, \quad \sum_{i=0}^{k} x_i = \sum_{j=0}^{l} y_j = 0,$$

which can be written as

$$\sum_{i=1}^{k} x_i = \sum_{j=1}^{l} y_j = 0.$$

And these equations (together with the equations of the ceiling which can be reduced by means of the graph of the framework) have a unique solution if and only if the given conditions in Theorem 1.2 are satisfied.

In the case of the 1×1 one-story building its matroid $M_B(1,1)$ will be very simple, because we can observe that if we put diagonal rods into any four squares of the building the framework will be rigid. The matrix of the matroid is the following:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{array}{l} \to \text{upper } X\text{-wall} \\ \to \text{lower } X\text{-wall} \\ \to \text{left-side } Y\text{-wall} \\ \to \text{right-side } Y\text{-wall} \\ \to \text{ceiling} \end{array}$$

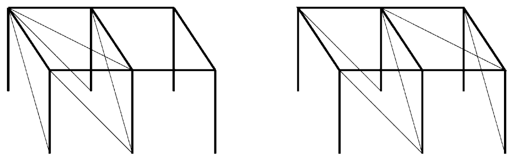

Fig. 5.

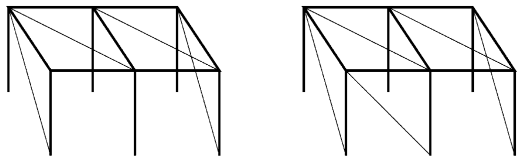

Fig. 6.

The row space matroid of this matrix is a graphic matroid which is isomorphic to the cycle matroid of the graph C_5

$$M_B(1,1) \simeq M(C_5).$$

But it is the only case when the matroid of the one-story building is graphic.

Theorem 2.3. *The matroid $M_B(k,l)$ of a $k \times l$ one-story building is binary if and only if $k = l = 1$.*

Proof. Welsh [6, p. 162] proved that a matroid is binary if and only if the symmetric difference of any pair of distinct circuits $C_1 \triangle C_2$ contains a circuit C. Now, we will show a counterexample in the matroid $M_B(1,2)$. Let us consider the two circuits of the building in Fig. 5. They are actually circuits because omitting any rod from them we will obtain a base of the 1×1 building which can be extended to a base of the 1×2 building. The symmetric difference of the circuits, and a base obtained from the symmetric difference can be seen in Fig. 6, so there is no circuit in the symmetric difference.

This construction proves the non-binarity in case of $M_B(2,1)$, and we can observe that if $k' \leqslant k$ and $l' \leqslant l$ then $M_B(k',l')$ is a minor of $M_B(k,l)$ (we can obtain it by deleting the unnecessary squares from the matroid). So none of the matroids $M_B(k,l)$ $(k+l > 2)$ can be binary. $\square$

We can state more about the representability of these matroids. Let us consider a $(1 \times l)$-sized one-story building and put the diagonals of all horizontal squares and the diagonal of a vertical square in one of the long vertical walls. Let all of the remaining squares in the long vertical walls be forbidden except one square in the unbraced wall. One can easily check that the matroid of the remaining squares is isomorphic to the uniform matroid $U_{l+2,2}$. (In Fig. 7 the braced vertical squares and the horizontal diagonals are marked by continuous line segments and the forbidden vertical squares

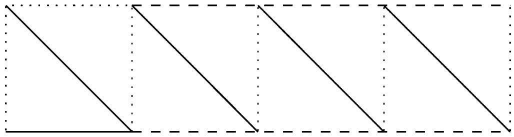

Fig. 7.

are marked by broken lines segments from above.) Bracing any two of the six dotted line segments leads to a minimum rigid system.

Hence we obtain the following theorem:

Theorem 2.4. *For every given finite field F_q there exists a number l_0 so that the matroid of the $(1 \times l)$-sized one-story buildings ($l \geqslant l_0$) cannot be representable over the field F_q.*

Proof. The previous construction showed us that the matroid of the $(1 \times l)$-sized one-story building contains a minor which is isomorphic to $U_{l+2,2}$. But the uniform matroid $U_{n,2}$ cannot be represented over F_q if $q < n-1$. □

Conclusion: The matroid of a $(k \times l)$-sized one-story building can be represented over the finite field F_q only if $k,\ l \leqslant q-1$.

3. Multi-story buildings

A t-story building consists of t pieces of $k \times l$ square grids, the first connected to the ground by vertical rods from each joint, the second connected to the first from above by vertical rods from each joint and so on. So these are t pieces of one-story buildings on one another. Hence, a necessary condition can be easily obtained: in each level there must be at least three braced vertical walls which are not all parallel.

Let us suppose that the vertical walls in each floor are sufficiently braced. Due to the vertical rods the infinitesimal motion of the joints, similar to the case of one-story buildings, can only be horizontal, so we have to prevent the horizontal motions. If we put a diagonal rod into a vertical wall it prevents the shear of this wall within its plane. So, if we have two crossing vertical walls braced then the vertical rod in the intersection remains vertical, which means that the horizontal (the only possible) motion of the two joints of this rod will be the same.

The corridors of the pth horizontal grid (from the bottom) will be $X_1^p, X_2^p, \ldots$ and $Y_1^p, Y_2^p, \ldots$, respectively, and their rotation will be marked with the corresponding lower case letters. As we mentioned in each level of the building there are at least one X-wall and at least one Y-wall braced (together at least three walls). Then it is easy to observe that if there are two braced X-walls (the uth and the vth) in the pth floor then the sums of the rotations of the X-corridors of the $(p-1)$th and pth floor between the two walls must be equal:

$$\sum_{i=u}^{v-1} x_i^{p-1} = \sum_{j=u}^{v-1} x_j^p. \tag{4}$$

Similar to the case of the one-story buildings we can construct a new system of equations as the effect of the diagonal rods by introducing the new variables $x_0^1, x_0^2, \dots, x_0^t, y_0^1, y_0^2, \dots, y_0^t$. The equation that corresponds to the diagonal rod in the rth X-wall of the pth floor is

$$\sum_{i=0}^{r-1} x_i^{p-1} = \sum_{j=0}^{r-1} x_j^p \tag{5}$$

and a similar equation with y_i^{p-1}'s and y_j^p's corresponds to the diagonal rod in the rth Y-wall of the pth floor. The equation corresponding to the diagonal rod in the intersection of the uth X-corridor and the vth Y-corridor of the pth floor:

$$x_u^p - y_v^p = 0. \tag{6}$$

This new system of equations of a $k \times l$ sized t-story building, by analogy with the extended one-story building (see Fig. 4), can be corresponded to a $(k+1)\times(l+1)$-sized t-story building where the external walls of the new corridors are braced in each level. So, a system of diagonal rods makes the original t-story building rigid if and only if it makes the extended building rigid.

In the case of a $k \times l \times t$ building, which is sufficiently braced, we have $t(k+l+2)$ variables and as many equations as the number of the diagonal rods in the building. The building is rigid if and only if the system of equations of types (5) and (6) has the zero vector as the only solution.

Theorem 3.1. *A $(k \times l)$-sized t-story building is infinitesimally rigid if and only if*

(i) *there are at least three vertical walls braced in each floor and these are not all parallel, and*

(ii) *the system of equations obtained from the rods has the zero vector as the only solution.*

Remark 4. Similar to the case of the one-story buildings the minimal sets of squares whose diagonals make the building rigid form the base set of a representable matroid, and a convenient representation is the matrix, constructed from the coefficient vectors of the equations, corresponding to the diagonals of the squares. It means that choosing an appropriate minimal system of diagonal rods we can take our preferences about braced squares into consideration.

Let us observe that this theorem gives us much more possibilities to rigidify the whole building than to make the levels of the building rigid one by one. For example, let us consider a (2×2)-sized 2-story building. If we make the upper ceiling rigid as a planar grid (with $2+2-1=3$ diagonal rods forming a tree) and put diagonal rods

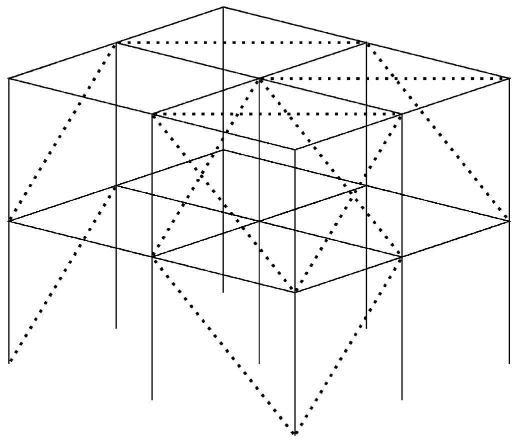

Fig. 8.

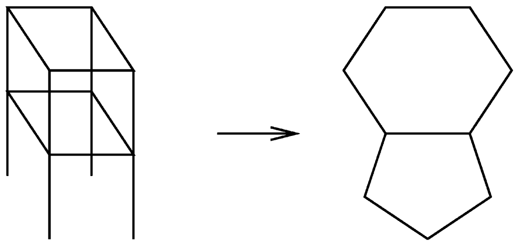

Fig. 9.

into all of the vertical walls of the second level then bracing any three of the vertical walls of the first level (of course not all parallel) will make the whole building rigid. In this system there are only 3 diagonal rods in the first level and 9 rods in the second level (see Fig. 8).

We showed that the matroid of a $(k \times l)$-sized one-story building cannot be represented over the field F_q if k or l is greater than $q-1$. The matroids of the one-story buildings are minors of the matroids of higher buildings, so this observation holds for the $(k \times l)$-sized t-story building, as well. But what about $1 \times 1 \times t$ buildings? We prove that the matroid in these special cases will be graphic.

One can observe that in the case of the $1 \times 1 \times 2$ building there are three circuits in the matroid on the squares: if we brace all squares of an elementary cube (it means two circuits with length of 5 and 6, respectively), or we brace all of the vertical squares and the top of the building (it is a circuit of length 9). So the matroid of this building is graphic, the corresponding graph is shown in Fig. 9.

After these observations it is easy to see that in the case of a $1 \times 1 \times t$ building the circuits are the sets of squares which cover the surface of elementary cubes and the sets

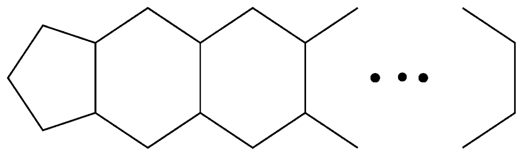

Fig. 10.

which can be obtained as the symmetric difference of some of these sets corresponding to adjacent cubes. So we can state the following

Theorem 3.2. *The matroid of a* $1 \times 1 \times t$ *building is always graphic and the corresponding graph is a "chain" of a pentagon (from the first floor) and* $t-1$ *pieces of hexagons, like in* Fig. 10.

We have to mention that all theorems in this paper remain true if we consider the buildings as grids of rectangles connected to each other and to the ground by vertical rods of unique length at every joint. The only thing we have to do is to substitute x_i^p/(the width of the ith X-corridors) for x_i^p's. We need it to solve the equations, but the representations of the matroids will not be changed.

Acknowledgements

Research partly supported by grants No. OTKA 30122 of the Hungarian National Science Foundation and by grant No. FKFP 409/1997 of the Hungarian Ministry of Education.

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